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FIXED POINT RESULTS IN TRIPLE CONTROLLED METRIC-LIKE SPACES

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ABSTRACT

In this paper, we prove fixed point results of Kannan and Reich interpolative contraction in triple contraction metric like spaces. Moreover, examples have also been provided to underpin and exemplify the results.

KEYWORDS: *Interpolative Contraction, Fixed Point, Triple Controlled Metric Space, Triples Controlled Metric-Like Spaces*

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INTRODUCTION

Fixed point theory is indeed a rich and expansive area of study that has significant applications across various fields, such as mathematical sciences, engineering, and computer science. The foundational work of Banach [1], particularly his introduction of the Banach contraction principle, laid the groundwork for understanding fixed points in complete metric spaces.

The Banach contraction principle states that if a mapping Ţ on a complete metric space (Ḫ, Ᵽ) is a contraction mapping (i.e., there exists a constant k ε (0,1) such that Ᵽ(Tσ, Tρ) ≤ k Ᵽ(σ, ρ), for all σ, ρ ε Ḫ then Ţ has a unique fixed point σ* such that Ţσ* = σ*). This theorem not only serves as a fundamental result in the analysis of dynamical systems but also finds utility in the convergence of iterative methods, making it indispensable in numerical methods and algorithm design.SinceBanach's time, the theory has undergone extensive generalization and refinement. Researchers have explored conditions under which fixed points exist in various types of spaces, including topological spaces, partially ordered sets, and more generalized metric spaces**.** (see-[1-27]).

The increasing interest in b-metric spaces has been marked by significant contributions from researchers such as Bakhtin [2] and Czerwik [3], which have laid the groundwork for numerous fixed-point theorems. Subsequently, Kamran et al.[5] introduced the concept of extended b-metric spaces, which emphasizes a nuanced control of the triangle inequality, moving beyond the traditional reliance on control functions in contractive conditions .Building on these foundational concepts, Abdeljawad et al.[4] presented double controlled metric spaces, leading to a wealth of new fixed-point results. The introduction of triple controlled metric spaces by Tasneem et al.[26] further enriched the study of these mathematical structures, resulting in additional theorems and a deeper comprehension of the layered relationships in metric-like spaces.

Harandi [22] proposal of metric-like spaces serves to generalize traditional metric spaces, while Mlaiki [19] work on controlled metric-like spaces makes allowances for non-zero self-distance, expanding the conceptual scope of these spaces. This evolution has led to the formulation of triple controlled metric-like spaces (see- [25]), which aim to foster the development of new fixed-point theorems and demonstrate practical applications through concrete examples.

In a contemporary context, Singh et al.[20] have introduced various new interpolative contractions—such as the (λ, a)-interpolative Kannan contraction, the (λ, a, b)-interpolative Kannan contraction, and the (λ, a, b, c)-interpolative Reich contraction within complete controlled metric spaces.

This article extends the existing body of knowledge by providing fixed-point results for the Kannan and Reich interpolating contractions within the framework of triple controlled metric-like spaces. Accompanying this theoretical exposition, practical examples are offered to demonstrate and elucidate the findings, highlighting the relevance and practical implications of these advanced mathematical frameworks in both theoretical exploration and applied scenarios.

PRELIMINARIE

The definitions provided describe various structures related to generalized metric spaces, each exhibiting certain controlled behaviors governed by specific functions. Below, I provide a summary of the definitions:

Summary of Definitions

Definition [2] (b-Metric Space)

Let $Ⱨ \neq \phi$ ans $s \geq 1$.A function $Ꝑ_b : Ⱨ x Ⱨ \rightarrow [0,+\infty)$ is called a b- metric if it satisfies:

- Non-negativity:$Ꝑ_b(\rho, \sigma) \geq 0$,
- Identity of indiscernibles: $Ꝑ_b(\rho, \sigma) = 0$ if and only if $\rho = \sigma$,
- Symmetry: $Ꝑ_b(\rho, \sigma) = Ꝑ_b(\sigma, \rho)$,
- Boundedness:$Ꝑ_b(\rho, \sigma) \leq s[Ꝑ_b(\rho, \delta) + Ꝑ_b(\delta, \sigma)]$, for all $\rho, \sigma, \delta \, \varepsilon \, Ⱨ$.

  The pair $(Ⱨ, Ꝑ_b)$ is a b-metric space.

Definition [5] (Extended b-Metric Space)

Let $Ⱨ \neq \phi$ and $\alpha: Ⱨ x Ⱨ \rightarrow [1,+\infty)$ is a function. Let $Ꝑ_{eb} : Ⱨ x Ⱨ \rightarrow [0,+\infty)$ be a function is an extended b- metric if:

- Non-negativity.
- Identity of indiscernibles.
- Symmetry.
- Extended boundedness.$Ꝑ_{eb}(\rho, \sigma) \leq \alpha(\rho, \sigma)[Ꝑ_{eb}(\rho, \delta) + Ꝑ_{eb}(\delta, \sigma)]$, for all $\rho, \sigma, \delta \, \varepsilon \, Ⱨ$.

  The pair $(Ⱨ, Ꝑ_{eb})$ is an extended b-metric space.

Definition [6] (Controlled Metric Space)

Let Ӈ≠ ϕ and α: Ӈ x Ӈ → [1,+ ∞) is a function. Let $Ᵽ_c$: Ӈx Ӈ → [0,+ ∞) be a function is a controlled metric if:

- Non-negativity.
- Identity of indiscernibles.
- Symmetry.
- Controlled boundedness.$Ᵽ_c$(ρ, σ) ≤α(ρ, δ) $Ᵽ_c$(ρ, δ) +α(δ, σ) $Ᵽ_c$(δ, σ)], for all ρ, σ, δ ε Ӈ.

  The pair (Ӈ, $Ᵽ_c$) is a controlled b-metric space.

Definition [7] (Double Controlled Metric Space)

Let Ӈ≠ ϕ and α,β: Ӈ x Ӈ → [1,+ ∞) is a function. Let $Ᵽ_{dc}$: Ӈx Ӈ → [0,+ ∞) be a function is a double controlled metric if:

- Non-negativity.
- Identity of indiscernibles.
- Symmetry.
- Doublecontrolled boundedness.$Ᵽ_{dc}$(ρ, σ) ≤α(ρ, δ) $Ᵽ_{dc}$(ρ, δ) +β(δ, σ) $Ᵽ_{dc}$(δ, σ)], for all ρ, σ, δ ε Ӈ.

  The pair (Ӈ, $Ᵽ_{dc}$) is a double controlled b-metric space.

Definition [24] (Triple Controlled Metric Space)

Let Ӈ≠ ϕ and α,β,γ: Ӈ x Ӈ → [1,+ ∞) is a function. Let $Ᵽ_{tc}$: Ӈx Ӈ → [0,+ ∞) be a function is a triple controlled metric if:

- Non-negativity.
- Identity of indiscernibles.
- Symmetry.
- Triple controlled boundedness.$Ᵽ_{tc}$(ρ, σ) ≤α(ρ, δ) $Ᵽ_{tc}$(ρ, δ) +β(δ, ∂) $Ᵽ_{tc}$(δ, ∂) +α(∂, σ) $Ᵽ_{tc}$(∂, σ)], for all ρ, σ, δ,∂ ε Ӈ.

  The pair (Ӈ, $Ᵽ_{tc}$) is a triple controlled metric space.

Definition 2.6[19] (Controlled Metric - Like Space)

Let Ӈ≠ ϕ and α: Ӈ x Ӈ → [1,+ ∞) is a function. Let $Ᵽ_{cl}$: Ӈx Ӈ → [0,+ ∞) be a function is an controlled metric-like if:

- $Ᵽ_{cl}$(ρ, σ) ≥ 0,
- $Ᵽ_{cl}$(ρ, σ) = 0 implies ρ = σ,
- $Ᵽ_{cl}$(ρ, σ) = $Ᵽ_{cl}$(σ, ρ),
- $Ᵽ_{cl}$(ρ, σ) ≤ α(ρ, δ) $Ᵽ_{cl}$(ρ, δ) + α(δ,σ) $Ᵽ_{cl}$(δ, σ), for all ρ, σ, δ ε Ӈ.

  The pair (Ӈ, $Ᵽ_c$) is called a controlled metric-like space.

Definition [21] (Double Controlled Metric - Like Space)

Let Ꜧ$\neq \phi$ and α,β: Ꜧ x Ꜧ $\rightarrow [1,+\infty)$ is a function. Let Ᵽ$_{dcl}$: Ꜧx Ꜧ $\rightarrow [0,+\infty)$ be a function is an double controlled metric-like if:

- Non-negativity.
- Identity of indiscernibles (implies condition).
- Symmetry.
- Doublecontrolled boundedness.Ᵽ$_{dcl}(\rho, \sigma) \leq \alpha(\rho, \delta)$ Ᵽ$_{dcl}(\rho, \delta) + \beta(\delta, \sigma)$ Ᵽ$_{dcl}(\delta, \sigma)$], for all $\rho, \sigma, \delta \in$ Ꜧ.

  The pair (Ꜧ, Ᵽ$_{dcl}$) is a double controlled metric- like space.

Definition 2.8[25] (Triple Controlled Metric - Like Space)

Let Ꜧ$\neq \phi$ and α,β,γ: Ꜧ x Ꜧ $\rightarrow [1,+\infty)$ is a function. Let Ᵽ$_{tcl}$: Ꜧx Ꜧ $\rightarrow [0,+\infty)$ be a function is an triple controlled metric-like if:

- Non-negativity.
- Identity of indiscernibles (implies condition).
- Symmetry.
- Doublecontrolled boundedness.Ᵽ$_{tcl}(\rho, \sigma) \leq \alpha(\rho, \delta)$ Ᵽ$_{tcl}(\rho, \delta) + \beta(\delta, \partial)$ Ᵽ$_{tcl}(\delta, \partial) + \alpha(\partial, \sigma)$ Ᵽ$_{tcl}(\partial, \sigma)$], for all $\rho, \sigma, \delta, \partial \in$ Ꜧ. The pair (Ꜧ, Ᵽ$_{tcl}$) is a triple controlled metric- like space.

A triple controlled metric- type space is also double controlled metric – like space in general. The converse is not true in general.

Example [25]

Let Ꜧ = { 0,1,2,3}. Condider the triple controlled metric-like Ᵽ$_{tcl}(\rho,\sigma)$: ꜦxꜦ$\rightarrow[0,\infty)$ defined by

Table 1

Ᵽ$_{tcl}(\rho,\sigma)$	0	1	2	3
0	1	1	1	3
1	1	1	2	1
2	1	2	2	1/3
3	3	1	1/3	0

Taking α, β, γ: ꜦxꜦ $\rightarrow[1,\infty)$ to be symmetric and defined by

Table 2

$\alpha(\rho,\sigma)$	0	1	2	3
0	1	1	1	4/3
1	1	1	1	3/2
2	1	1	1	3
3	4/3	3/2	3	1

Table 3

β(ρ,σ)	0	1	2	3
0	1	1	2	1
1	1	1	3/2	1
2	2	3/2	1	4
3	1	1	4	1

Table 4

γ(ρ,σ)	0	1	2	3
0	1	1	1	1
1	1	1	2	1
2	1	2	1	3
3	1	1	3	1

One can easily show that (Ʉ,$Ꝑ_{tcl}$) is a triple controlled metric –like space rather than a triple controlled metric space $Ꝑ_{tcl}(1,1)= 1 \neq 0$.

1. $Ꝑ_{tcl}(0,3)=3 > \alpha(0,1)Ꝑ_{tcl}(0,1)+\beta(1,3)\, Ꝑ_{tcl}(1,3) = 1x1 +1x1 = 2$.

Hence (Ʉ,$Ꝑ_{tcl}$) is not double controlled metric space not double controlled metric like space.

2. $Ꝑ_{tcl}(0,3)=3 > \alpha(0,1)Ꝑ_{tcl}(0,1)+\alpha(1,3)\, Ꝑ_{tcl}(1,3) = 1x1 +3/2x1 = 2.5$

Hence (Ʉ,$Ꝑ_{tcl}$) is not controlled metric space not controlled metric like space.

3. $Ꝑ_{tcl}(0,3)=3 > \alpha(0,3)[Ꝑ_{tcl}(0,1)+ Ꝑ_{tcl}(1,3)] = 4/3[1 +1]= 8/3$.

Hence (Ʉ,$Ꝑ_{tcl}$) is not extended b-metric space not extended b-metric like space.

Definition [25] (Convergence and Cauchy Sequences)

Let (Ƕ,$Ꝑ_{tcl}$) be a triple controlled metric -like space by one or two functions.

- **Convergence**: A sequence $\{\sigma_n\}$ is convergent to some σ in Ʉ, if for each positive ε, there is some integer Z_ε such that $Ꝑ_{tcl}(\sigma_n,\sigma) < \varepsilon$ for each $n \geq Z_\varepsilon$.

  It is written as $\lim_{n\to\infty} \sigma_n = \sigma$.

- **Cauchy Sequence**: A sequence $\{\sigma_n\}$ is said Cauchy, if for every ε > 0, $Ꝑ_{tcl}(\sigma_n,\sigma_m) < \varepsilon$ for all $m,n \geq Z_\varepsilon$, where Z_ε is some integer.

- **Completeness**: (Ƕ,$Ꝑ_{tcl}$) is said complete if every Cauchy sequence is convergent.

Definition [25] (Continuity of Self- Mapping)

Let (Ƕ,$Ꝑ_{tcl}$) be a triple double controlled metric like space by one or two functions for σ ε Ʉ and l > 0.

- **Open Ball**: Define the open ball centered at σ with radius l as: $B(\sigma, l) = [\, y\, \varepsilon Ʉ,\, Ꝑ_{tcl}(\sigma, y) < l\, \}$.

- **Continuity of Self- Mapping**: A self mapping Ţ on Ʉ is said to be continuous at σ in Ʉ if for all δ > 0, there exists l > 0 such that Ţ(B(σ, l)) ⊆ B(Ţσ,δ).

  This means that if you take point close to σ, there image under Ţ will be close to Ţσ.

Finally, If Ţ is continuous at σ in (Ɉ, Ꝑ$_{tcl}$) , then the convergence of the sequence $\sigma_n \rightarrow \sigma$ implies that $Ţ\sigma_n \rightarrow Ţ\sigma$ when $n \rightarrow \infty$.

- **Lemma** 2.1 [25] Let (Ꞻ,Ꝑ$_{tcl}$) be a triple controlled metric like space and assume a sequence $\{\sigma_n\}$ in Ɉ. Then $\{\sigma_n\}$ is Cauchy sequence, then Ꝑ$_{tcl}$ $(\sigma_n,\sigma_m) \rightarrow 0$ as n, m $\rightarrow\infty$where n,m ε N.
- **Lemma** 2.2 [25] Let (Ꞻ,Ꝑ$_{tcl}$) be a triple controlled metric like space. Then a sequence $\{\sigma_n\}$ in Ɉ is Cauchy sequence, such that $\sigma_n \neq \sigma_m$, whenever $m \neq n$. Then $\{\sigma_n\}$ converges to at most one point.
- **Lemma** 2.3 [25] Let (Ꞻ,Ꝑ$_{tcl}$) be a triple controlled metric like space and assume a sequence $\{\sigma_n\}$ in Ɉ. Then $\{\sigma_n\}$ is converges to σ, then Ꝑ$_{tcl}$ $(\sigma_n,\sigma) \rightarrow 0$ as $n\rightarrow\infty$.
- **Lemma** 2.3 [25] For a given triple controlled space (Ɉ, Ꝑ$_{tcl}$) , the triple controlled metric like function Ꝑ$_{tcl}$: Ɉ x Ɉ → C is continuous,
- **Lemma** 2.4 [25] Let (Ꞻ,Ꝑ$_{tcl}$) be a triple controlled metric like space. Limit of every convergent sequence in Ɉ is unique, if thr functional Ꝑ$_{tcl}$: Ɉ x Ɉ →Ɉ is continuous.

RESULT

In this section, we introduce several types of interpolative contractions defined in the context of triple controlled metric-like spaces. Specifically, we explore the (λ, a)-interpolative Kannan contraction, the (λ, a, b)-interpolative Kannan contraction, and the (λ, a, b, c)-interpolative Reich contraction. We also provide theorems related to Kannan and Reich contractions, supported by examples.

Definition ((λ, a)-interpolative Kannan Contraction)

A self mapping Ţ on a set Ɉ is termed a (k, a) – interpolative Kannan contraction, if there exist parameters k ε [0,1). a ε (0,1) such that

$$Ꝑ_{tcl} (Ţ\sigma, Ţ\rho) \leq k (Ꝑ_{tcl} (\sigma, Ţ\sigma))^{a} (Ꝑ_{tcl} (\rho, Ţ\rho))^{1-a} \qquad 3.1$$

for all σ, ρ ε Ɉ , with $\sigma \neq \rho$.

Definition ((λ, a,b)-interpolative Kannan Contraction)

A self mapping Ţ on a set Ɉ is termed a (k, a,b) – interpolative Kannan contraction, if there exist parameters, k ε [0,1), a, b ε (0,1) , a + b < 1 such that

$$Ꝑ_{tcl} (Ţ\sigma, Ţ\rho) \leq k (Ꝑ_{tcl} (\sigma, Ţ\sigma))^{a} (Ꝑ_{tcl} (\rho, Ţ\rho))^{b} \qquad 3.2$$

for all σ, ρ ε Ɉ , with $\sigma \neq \rho$.

Definition ((λ, a,b,c)-interpolative Reich Contraction)

A self mapping Ţ on a set Ɉ is termed a (k, a,b,c) – interpolative Reich contraction, if there exist parameters k ε [0,1), a, b, c ε (0,1), a + b + c < 1 such that

$$Ꝑ_{tcl} (Ţ\sigma, Ţ\rho) \leq k (Ꝑ_{tcl} (\sigma, \rho))^{a} (Ꝑ_{tcl} (\sigma, Ţ\sigma))^{b} (Ꝑ_{tcl} (\rho, Ţ\rho))^{c} \qquad 3.3$$

for all σ, ρ ε Ɉ , with $\sigma \neq \rho$.

Theorem 3.1

Let (Ή,Ꝑ$_{tcl}$) be a complete triple controlled metric like space.Aself mapping on a set Ӈ is termed a (k, a) – interpolative Kannan contraction. For σ_0 ε Ӈ, take $\sigma_n = Ţ^n\sigma_0$. Assume that

$$\sup_{m\geq 1}\lim_{i\to\infty}\gamma(\sigma_i, \sigma_{im}) [\alpha(\sigma_{i+1}, \sigma_{i+2})+ k\,\beta(\sigma_{i+2}, \sigma_{i+3})]/ [\alpha(\sigma_i, \sigma_{i+1})+ k\beta(\sigma_{i+1}, \sigma_{i+2})< 1/k \quad 3.4$$

We assume that ,for σ ε Ӈ, we have $\lim_{n\to\infty}\Phi(\sigma_n, \sigma)$, $\lim_{n\to\infty}\Phi(\sigma, \sigma_n)$,$\lim_{n\to\infty}\Phi(\sigma_n, Ţ\sigma)$,$\lim_{n\to\infty}\Phi(Ţ\sigma, \sigma_n)$, $\lim_{n,m\to\infty}\Phi(\sigma_m, \sigma_n)$

where Φ ε { α , β ,γ }, exist and finite for all n,m ε N, m≠n. .

Then Ţ has a fixed point.

Proof

Define a sequence $\{\sigma_n\}$ as $\sigma_{n+1} = Ţ\sigma_n$ for all n ε N. Suppose that $\sigma_{n+1} \neq \sigma_n$ for each n ε N. Thus, by 3.1, we have

$$Ꝑ_{tcl} (\sigma_n, \sigma_{n+1}) = Ꝑ_{tcl} (Ţ\sigma_{n-1} , Ţ\sigma_n) \leq k \ (Ꝑ_{tcl} (\sigma_{n-1} , Ţ\sigma_{n-1}))^a \ (Ꝑ_{tcl} (\sigma_n, Ţ\sigma_n))^{1-a}$$

$$= k (Ꝑ_{tcl} (\sigma_{n-1} , \sigma_n))^a \ (Ꝑ_{tcl} (\sigma_n, \sigma_{n+1}))^{1-a}$$

$$(Ꝑ_{tcl} (\sigma_n, \sigma_{n+1}))^a \leq k(Ꝑ_{tcl} (\sigma_{n-1} , \sigma_n))^a \quad 3.5$$

Since a < 1, we have

$$Ꝑ_{tcl} (\sigma_n, \sigma_{n+1}) \leq k^{1/a}(Ꝑ_{tcl} (\sigma_{n-1} , \sigma_n)) \leq k\, Ꝑ_{tcl} (\sigma_{n-1} , \sigma_n)$$

$$Ꝑ_{tcl} (\sigma_n, \sigma_{n+1}) \leq k\, Ꝑ_{tcl} (\sigma_{n-1} , \sigma_n) \leq k^2 Ꝑ_{tcl} (\sigma_{n-2} , \sigma_{n-1}) \leq k^3 Ꝑ_{tcl} (\sigma_{n-3} , \sigma_{n-2})\ldots \leq k^n Ꝑ_{tcl} (\sigma_0 , \sigma_1) \quad 3.6$$

Taking limit n→∞, we get $\lim_{,n\to\infty}Ꝑ_{tcl} (\sigma_n, \sigma_{n+1}) = 0$. 3.7

Similarly, $\lim_{,n\to\infty}Ꝑ_{tcl} (\sigma_{n+1}, \sigma_n) = 0$. 3.8

Thus, we two cases,

Case I. Suppose $\sigma_n \neq \sigma_m$ for all n,m ε N . Let $\sigma_n = \sigma_m$ for m= n+r which r>0, and $Ţ\sigma_n = Ţ\sigma_m$, then,

$$Ꝑ_{tcl} (\sigma_n, \sigma_{n+1}) = Ꝑ_{tcl} (\sigma_m, \sigma_{m+1}) = Ꝑ_{tcl} (Ţ\sigma_{m-1}, Ţ\sigma_m) \leq kꝐ_{tcl} (\sigma_{m-1}, \sigma_m)\leq k^2Ꝑ_{tcl} (\sigma_{m-1}, \sigma_m)\ldots\leq k^rꝐ_{tcl} (\sigma_m, \sigma_{m+1})$$

$$= k^rꝐ_{tcl} (\sigma_n, \sigma_{n+1})$$

$(1-k^r)Ꝑ_{tcl} (\sigma_n, \sigma_{n+1})\leq 0$, implies $Ꝑ_{tcl} (\sigma_n, Ţ\sigma_n) = 0$.

Similarly $Ꝑ_{tcl} (Ţ\sigma_n, \sigma_n) = 0$.Thus $\sigma_n$is a fixed point of Ţ.

Case II

Suppose $\sigma_n \neq \sigma_m$ for all n,m ε N. let n< m , and to show that $\{\sigma_n\}$ is a right Cauchy sequence we consider two subcases,

Subcase I

For all n,m ε N and n < m, let m=n+2p+1 with

p≥1.$Ꝑ_{tcl} (\sigma_n, \sigma_m) = Ꝑ_{tcl} (\sigma_n, \sigma_{n+2p+1}) \leq \alpha(\sigma_n, \sigma_{n+1}) \ Ꝑ_{tcl} (\sigma_n, \sigma_{n+1}) + \beta(\sigma_{n+1}, \sigma_{n+2})\, Ꝑ_{tcl} (\sigma_{n+1}, \sigma_{n+2}) + \gamma(\sigma_{n+2}, \sigma_{n+2p+1}) \ Ꝑ_{tcl} (\sigma_{n+2}, \sigma_{n+2p+1}) \leq \alpha(\sigma_n, \sigma_{n+1}) \ Ꝑ_{tcl} (\sigma_n, \sigma_{n+1}) +\beta(\sigma_{n+1}, \sigma_{n+2}) \ Ꝑ_{tcl} (\sigma_{n+1}, \sigma_{n+2}) + \gamma(\sigma_{n+2}, \sigma_{n+2p+1}) \{ \alpha(\sigma_{n+2}, \sigma_{n+3}) \ Ꝑ_{tcl} (\sigma_{n+2}, \sigma_{n+3}) + \beta(\sigma_{n+3}, \sigma_{n+4})\, Ꝑ_{tcl} (\sigma_{n+3}, \sigma_{n+4}) +\gamma(\sigma_{n+4}, \sigma_{n+2p+1}) \ Ꝑ_{tcl} (\sigma_{n+4}, \sigma_{n+2p+1}) \ \}$

$= \alpha(\sigma_n, \sigma_{n+1})\ \text{Ꝑ}_{tcl}(\sigma_n, \sigma_{n+1}) + \beta(\sigma_{n+1}, \sigma_{n+2})\ \text{Ꝑ}_{tcl}(\sigma_{n+1}, \sigma_{n+2}) + \gamma(\sigma_{n+2}, \sigma_{n+2p+1})\ \alpha(\sigma_{n+2}, \sigma_{n+3})\ \text{Ꝑ}_{tcl}(\sigma_{n+2}, \sigma_{n+3}) + \gamma(\sigma_{n+2}, \sigma_{n+2p+1})\ \beta(\sigma_{n+3}, \sigma_{n+4})\ \text{Ꝑ}_{tcl}(\sigma_{n+3}, \sigma_{n+4}) + \gamma(\sigma_{n+2}, \sigma_{n+2p+1})\ \gamma(\sigma_{n+4}, \sigma_{n+2p+1})\ \text{Ꝑ}_{tcl}(\sigma_{n+4}, \sigma_{n+2p+1})\ \}$

$\leq \alpha(\sigma_n, \sigma_{n+1})\ k^n\text{Ꝑ}_{tcl}(\sigma_0, \sigma_1) + \beta(\sigma_{n+1}, \sigma_{n+2})\ k^{n+1}\text{Ꝑ}_{tcl}(\sigma_0, \sigma_1) + \gamma(\sigma_{n+2}, \sigma_{n+2p+1})\ \alpha(\sigma_{n+2}, \sigma_{n+3})\ k^{n+2}\text{Ꝑ}_{tcl}(\sigma_0, \sigma_1) + \gamma(\sigma_{n+2}, \sigma_{n+2p+1})\ \beta(\sigma_{n+3}, \sigma_{n+4})\ k^{n+3}\text{Ꝑ}_{tcl}(\sigma_0, \sigma_1) + \gamma(\sigma_{n+2}, \sigma_{n+2p+1})\ \gamma(\sigma_{n+2p}, \sigma_{n+2p+1})\ k^{n+2p}\text{Ꝑ}_{tcl}(\sigma_0, \sigma_1)\} \leq \alpha(\sigma_n, \sigma_{n+1})\ k^n\text{Ꝑ}_{tcl}(\sigma_0, \sigma_1) + \beta(\sigma_{n+1}, \sigma_{n+2})\ k^{n+1}\text{Ꝑ}_{tcl}(\sigma_0, \sigma_1) + \sum_{i=n+2}^{n+2p} \prod_{j=n+2}^{i} \gamma(\sigma_j, \sigma_{n+2p+1})[\alpha(\sigma_i, \sigma_{i+1})\ k^i + \beta(\sigma_{i+1}, \sigma_{i+2})\ k^{i+1}]\text{Ꝑ}_{tcl}(\sigma_0, \sigma_1)$

$+ \prod_{j=n+2}^{n+2p} \gamma(\sigma_j, \sigma_{n+2p+1})\ k^{n+2p}\text{Ꝑ}_{tcl}(\sigma_0, \sigma_1)$ 3.7

Since, $\sup_{m\geq 1}\lim_{n\to\infty}\gamma(\sigma_j, \sigma_m)\ [\alpha(\sigma_{i+1}, \sigma_{i+2}) + k\ \beta(\sigma_{i+2}, \sigma_{i+3})\ ]/[\alpha(\sigma_i, \sigma_{i+1}) + k\ \beta(\sigma_{i+1}, \sigma_{i+2}) \leq 1/k$,

Then, the series

$\sum_{i=1}^{\infty} \prod_{j=1}^{i} \gamma(\sigma_j, \sigma_{n+2p+1})\ [\alpha(\sigma_i, \sigma_{i+1}) + k\ \beta(\sigma_{i+1}, \sigma_{i+2})\ ]k^i\text{Ꝑ}_{tcl}(\sigma_0, \sigma_1)$ 3.8

Converges by ratio test, implies that $\text{Ꝑ}_{tcl}(\sigma_n, \sigma_{n+2p+1})$ converges as $n\to\infty$.

Let $S_n = \sum_{i=1}^{\infty} \prod_{j=1}^{i} \gamma(\sigma_j, \sigma_{n+2p+1})\ [\alpha(\sigma_i, \sigma_{i+1}) + k\ \beta(\sigma_{i+1}, \sigma_{i+2})\ ]k^i\ \text{Ꝑ}_{tcl}(\sigma_0, \sigma_1)$ 3.9

Then equation 3.7 taks the following form

$\text{Ꝑ}_{tcl}(\sigma_n, \sigma_m) \leq [\alpha(\sigma_n, \sigma_{n+1})\ k^n + \beta(\sigma_{n+1}, \sigma_{n+2})\ ]k^{n+1} + S_{m-1} - S_{n+1}]\ \text{Ꝑ}_{tcl}(\sigma_0, \sigma_1)$

$+ \prod_{j=n+2}^{n+2p} \gamma(\sigma_j, \sigma_{n+2p+1})\ k^{n+2p}\text{Ꝑ}_{tcl}(\sigma_0, \sigma_1)$ 3.10

Taking the limit in 3.10 as, $n,m\to\infty$, implies $\lim_{m,n\to\infty}\text{Ꝑ}_{tcl}(\sigma_n, \sigma_m) = 0$. 3.11

Subcase II

When $m = n+2p$, first $p=1$, we have

$\text{Ꝑ}_{tcl}(\sigma_n, \sigma_m) = \text{Ꝑ}_{tcl}(\sigma_n, \sigma_{n+2}) = \text{Ꝑ}_{tcl}(\text{Ţ}\sigma_{n-1}, \text{Ţ}\sigma_{n+1}) \leq k\text{Ꝑ}_{tcl}(\sigma_{n-1}, \sigma_{n+1}) \leq k^2\text{Ꝑ}_{tcl}(\sigma_{n-2}, \sigma_n) \ldots \leq k^n\text{Ꝑ}_{tcl}(\sigma_0, \sigma_2)$

Implies that,

$\text{Ꝑ}_{tcl}(\sigma_n, \sigma_m) \to 0$ as $n\to\infty$.

When $p>1$, similarly to subcase I , we have

$\text{Ꝑ}_{tcl}(\sigma_n, \sigma_m) = \text{Ꝑ}_{tcl}(\sigma_n, \sigma_{n+2p}) \leq \alpha(\sigma_n, \sigma_{n+2})\ \text{Ꝑ}_{tcl}(\sigma_n, \sigma_{n+2}) + \beta(\sigma_{n+2}, \sigma_{n+3})\ \text{Ꝑ}_{tcl}(\sigma_{n+2}, \sigma_{n+3}) + \gamma(\sigma_{n+3}, \sigma_{n+2p})\ \text{Ꝑ}_{tcl}(\sigma_{n+3}, \sigma_{n+2p})$

$\leq \alpha(\sigma_n, \sigma_{n+2})\ \text{Ꝑ}_{tcl}(\sigma_n, \sigma_{n+2}) + \beta(\sigma_{n+2}, \sigma_{n+3})\ \text{Ꝑ}_{tcl}(\sigma_{n+2}, \sigma_{n+3}) + \gamma(\sigma_{n+3}, \sigma_{n+2p})\ \{\ \alpha(\sigma_{n+3}, \sigma_{n+4})\ \text{Ꝑ}_{tcl}(\sigma_{n+3}, \sigma_{n+4}) + \beta(\sigma_{n+4}, \sigma_{n+5})\ \text{Ꝑ}_{tcl}(\sigma_{n+4}, \sigma_{n+5}) + \gamma(\sigma_{n+5}, \sigma_{n+2p})\ \text{Ꝑ}_{tcl}(\sigma_{n+5}, \sigma_{n+2p})\ \}$

$= \alpha(\sigma_n, \sigma_{n+2})\ \text{Ꝑ}_{tcl}(\sigma_n, \sigma_{n+2}) + \beta(\sigma_{n+2}, \sigma_{n+3})\ \text{Ꝑ}_{tcl}(\sigma_{n+2}, \sigma_{n+3}) + \gamma(\sigma_{n+3}, \sigma_{n+2p})\ \alpha(\sigma_{n+3}, \sigma_{n+4})\ \text{Ꝑ}_{tcl}(\sigma_{n+3}, \sigma_{n+4}) + \gamma(\sigma_{n+3}, \sigma_{n+2p})\ \beta(\sigma_{n+4}, \sigma_{n+5})\ \text{Ꝑ}_{tcl}(\sigma_{n+4}, \sigma_{n+5}) + \gamma(\sigma_{n+3}, \sigma_{n+2p})\ \gamma(\sigma_{n+5}, \sigma_{n+2p})\ \text{Ꝑ}_{tcl}(\sigma_{n+5}, \sigma_{n+2p})\ \}$

$\leq \alpha(\sigma_n, \sigma_{n+2})\ k^n\text{Ꝑ}_{tcl}(\sigma_0, \sigma_2) + \beta(\sigma_{n+2}, \sigma_{n+3})\ k^{n+2}\text{Ꝑ}_{tcl}(\sigma_0, \sigma_1) + \sum_{i=n+3}^{n+2p-1} \prod_{j=n+3}^{i} \gamma(\sigma_j, \sigma_{n+2p})\ [\alpha(\sigma_i, \sigma_{i+1}) + k\beta(\sigma_{i+1}, \sigma_{i+2})]k^i\text{Ꝑ}_{tcl}(\sigma_0, \sigma_1) + \prod_{j=n+3}^{n+2p-1} \gamma(\sigma_j, \sigma_{n+2p})\ k^{n+2p-1}\text{Ꝑ}_{tcl}(\sigma_0, \sigma_1)$ 3.12

Since, $\sup_{m\geq 1}\lim_{i\to\infty}\gamma(\sigma_{i+1}, \sigma_m)\ [\alpha(\sigma_{i+1}, \sigma_{i+2}) + k\ \beta(\sigma_{i+2}, \sigma_{i+3})\ ]/[\alpha(\sigma_i, \sigma_{i+1}) + k\ \beta(\sigma_{i+1}, \sigma_{i+2})\ ] \leq 1/k$, Then,the series

$\sum_{i=1}^{\infty} \prod_{j=1}^{i} \gamma(\sigma_j, \sigma_{n+2p})\ [\alpha(\sigma_i, \sigma_{i+1}) + k\,\beta(\sigma_{i+1}, \sigma_{i+2})\ ]k^i\, Ꝑ_{tcl}\,(\sigma_0, \sigma_1)$ 3.13

Converges by ratio test, implies that $Ꝑ_{tcl}(\sigma_n, \sigma_{n+2p})$ converges as $n\to\infty$.

Let $S_t = \sum_{i=1}^{t} \prod_{j=1}^{i} \gamma(\sigma_j, \sigma_{n+2p})\ [\alpha(\sigma_i, \sigma_{i+1}) + k\,\beta(\sigma_{i+1}, \sigma_{i+2})\ ]k^i\, Ꝑ_{tcl}\,(\sigma_0, \sigma_1)$ 3.14

Then equation 3.12 takes the following form

$Ꝑ_{tcl}\,(\sigma_n, \sigma_m) \le \alpha(\sigma_n, \sigma_{n+2})\, k^n Ꝑ_{tcl}\,(\sigma_0, \sigma_2) + \ \beta(\sigma_{n+2}, \sigma_{n+3})\ ]k^{n+2}\, Ꝑ_{tcl}\,(\sigma_0, \sigma_1) + (S_{m-1} - S_{n+2})\, Ꝑ_{tcl}\,(\sigma_0, \sigma_1)$

$+ \prod_{j=n+3}^{n+2p-1} \gamma(\sigma_j, \sigma_{n+2p})\ k^{n+2p-1} Ꝑ_{tcl}\,(\sigma_0, \sigma_1)$ 3.15

Taking the limit in 3.10 as, $n,m\to\infty$, implies $\lim_{m,n\to\infty} Ꝑ_{tcl}\,(\sigma_m, \sigma_n) = 0$. 3.16

Thus by subcase I and subcase II, we get $\{\sigma_n\}$ is a right Cauchy sequence. Similarly we prove same as left Cauchy sequence. Since $(Ӈ, Ꝑ_{tcl})$ is complete triple controlled metric – like space. So, $\{\sigma_n\}$ converges to $\sigma^* \in Ӈ$. Thus $\lim_{n\to\infty} Ꝑ_{tcl}\,(\sigma_n, \sigma^*) = \lim_{n\to\infty} Ꝑ_{tcl}\,(\sigma^*, \sigma_n) = \lim_{m,n\to\infty} Ꝑ_{tcl}\,(\sigma_m, \sigma_n) = \lim_{m,n\to\infty} Ꝑ_{tcl}\,(\sigma_n, \sigma_m) = 0$. 3.17

Existance of Fixed Point

Now, let σ^* is a fixed point of Ţ. Suppose that $Ꝑ_{tcl}(\sigma^*, Ţ\sigma^*) > 0$ and $Ꝑ_{tcl}(Ţ\sigma^*, \sigma^*) > 0$. Then we get

$Ꝑ_{tcl}(Ţ\sigma^*, \sigma^*) \le \alpha(Ţ\sigma^*, Ţ\sigma_n) Ꝑ_{tcl}(Ţ\sigma^*, Ţ\,\sigma_n) + \beta(Ţ\,\sigma_n, \sigma_n) Ꝑ_{tcl}(Ţ\,\sigma_n, \sigma_n) + \gamma(\sigma_n, \sigma^*) Ꝑ_{tcl}(\sigma_n, \sigma^*)$

$\le \alpha(Ţ\sigma^*, \sigma_{n+1}) k Ꝑ_{tcl}(\sigma^*, \sigma_n) + \beta(\sigma_{n+1}, \sigma_n) Ꝑ_{tcl}(\sigma_{n+1}, \sigma_n) + \gamma(\sigma_n, \sigma^*) Ꝑ_{tcl}(\sigma_n, \sigma^*)$ and

$Ꝑ_{tcl}(\sigma^*, Ţ\sigma^*) \le \alpha(\sigma^*, \sigma_n) Ꝑ_{tcl}(\sigma^*, \sigma_n) + \beta(\sigma_n, Ţ\sigma_n) Ꝑ_{tcl}(\sigma_n, Ţ\sigma_n) + \gamma(Ţ\sigma_n, Ţ\sigma^*) Ꝑ_{tcl}(Ţ\sigma_n, Ţ\sigma^*)$

$\le \alpha(\sigma^*, \sigma_n) Ꝑ_{tcl}(\sigma^*, \sigma_n) + \beta(\sigma_n, \sigma_{n+1}) Ꝑ_{tcl}(\sigma_n, \sigma_{n+1}) + \gamma(\sigma_n, \sigma^*) k Ꝑ_{tcl}(\sigma_n, \sigma^*)$

Taking $m,n\to\infty$ and using 2.5,2.6, 2.17, we get $Ꝑ_{tcl}(Ţ\sigma^*, \sigma^*) = Ꝑ_{tcl}(\sigma^*, Ţ\sigma^*) = 0$. Implies

$Ţ\sigma^* = \sigma^*$ is a fixed point of Ţ.

Uniqueness of Fixed Point

Now, we prove the uniqueness of σ^*. Let ρ^* be another fixed point of Ţ inӇ, then by 3.1, we have

$Ꝑ_{cvdcl}\,(\sigma^*, \rho^*) = Ꝑ_{cvdcl}\,(Ţ\sigma^*, Ţ\rho^*) \le k\,[Ꝑ_{cvdcl}\,(\sigma^*, Ţ\sigma^*)\,]^a\,[Ꝑ_{cvdcl}\,(\rho^*, Ţ\rho^*)]^{1-a} = 0.$

Implies, $\sigma^* = \rho^*$. Complete the proof.

Example

LetӇ = {0,1,2,3}. Condider the triple controlled metric-like $Ꝑ_{tcl}(\rho,\sigma)$: ӇxӇ→[0,∞) defined by

Table 5

$Ꝑ_{tcl}(\rho,\sigma)$	0	1	2	3
0	1	1	1	3
1	1	1	2	1
2	1	2	2	1/3
3	3	1	1/3	0

Taking α, β, γ: ЊxЊ →[1,∞) to be symmetric and defined by

Table 6

α(ρ,σ)	0	1	2	3
0	1	1	1	4/3
1	1	1	1	3/2
2	1	1	1	3
3	4/3	3/2	3	1

Table 7

β(ρ,σ)	0	1	2	3
0	1	1	2	1
1	1	1	3/2	1
2	2	3/2	1	4
3	1	1	4	1

Table 8

γ(ρ,σ)	0	1	2	3
0	1	1	1	1
1	1	1	2	1
2	1	2	1	3
3	1	1	3	1

Let a mapping Ț: Њ→Њ, such that, Ț0= Ț1 =Ț2 = Ț3 =3, next, we will verify the condition 3.1:

Case 1

When σ = 0, ρ = 1,

$Ꝑ_{tcl}(Țσ,Țρ) = Ꝑ_{tcl}(Ț0,Ț1) = Ꝑ_{tcl}(3,3)= 0 \le k\ Ꝑ_{tcl}(0,Ț0)^a\ Ꝑ_{tcl}(1,Ț1)^{1-a}= k\ [Ꝑ_{tcl}(0,3)]^a[Ꝑ_{tcl}(1,3)]^{1-a} = k\ 3^a.\ 1^{1-a} = k.3^a\le .k[\ Ꝑ_{tcl}(σ,Țσ)]^a\ [Ꝑ_{tcl}(ρ,Țρ)]^{1-a}$

Case 2

When σ = 0, ρ = 2,

$Ꝑ_{tcl}(Țσ,Țρ) = Ꝑ_{tcl}(Ț0,Ț2) = Ꝑ_{tcl}(3,3)= 0 \le k\ Ꝑ_{tcl}(0,Ț0)^a\ Ꝑ_{tcl}(2,Ț2)^{1-a} = k\ [Ꝑ_{tcl}(0,3)]^a[Ꝑ_{tcl}(2,3)]^{1-a} = k\ 3^a\ (1/3)^{1-a} \le k\ [Ꝑ_{tcl}(σ,Țσ)]^a\ [Ꝑ_{tcl}(ρ,Țρ)]^{1-a}$

Case 3

When σ = 0, ρ = 3,

$Ꝑ_{tcl}(Țσ,Țρ) = Ꝑ_{tcl}(Ț0,Ț3) = Ꝑ_{tcl}(3,3)= 0 \le k\ Ꝑ_{tcl}(0,Ț0)^a\ Ꝑ_{tcl}(3,Ț3)^{1-a} = k\ [Ꝑ_{tcl}(0,3)]^a[Ꝑ_{tcl}(3,3)]^{1-a} = 0 \le k[\ Ꝑ_{tcl}(σ,Țσ)]^a\ [Ꝑ_{tcl}(ρ,Țρ)]^{1-a}$

Case 3

When σ = 0, ρ = 0,

$Ꝑ_{tcl}(Țσ,Țρ) = Ꝑ_{tcl}(Ț0,Ț0) = Ꝑ_{tcl}(3,3)= 0 \le k\ Ꝑ_{tcl}(0,Ț0)^a\ Ꝑ_{tcl}(0,Ț0)^{1-a} = k\ [Ꝑ_{tcl}(0,3)]^a[Ꝑ_{tcl}(0,3)]^{1-a} = k\ 3^a.\ 3^{1-a}\le k[\ Ꝑ_{tcl}(σ,Țσ)]^a\ [Ꝑ_{tcl}(ρ,Țρ)]^{1-a}$

Case 5

When $\sigma = 1$, $\rho = 0$,

$$Ꝑ_{tcl}(Ţ\sigma,Ţ\rho) = Ꝑ_{tcl}(Ţ1,Ţ0) = Ꝑ_{tcl}(3,3) = 0 \leq k\, Ꝑ_{tcl}(1,Ţ1)^{a}\, Ꝑ_{tcl}(0,Ţ0)^{1-a} = k\, [Ꝑ_{tcl}(1,3)]^{a}[Ꝑ_{tcl}(0,3)]^{1-a} = k\, 1^{a}.\, 3^{1-a} = k.3^{1-a} \leq k[\, Ꝑ_{tcl}(\sigma,Ţ\sigma)]^{a}\, [Ꝑ_{tcl}(\rho,Ţ\rho)]^{1-a}$$

Case 6

When $\sigma = 1$, $\rho = 2$,

$$Ꝑ_{tcl}(Ţ\sigma,Ţ\rho) = Ꝑ_{tcl}(Ţ1,Ţ2) = Ꝑ_{tcl}(3,3) = 0 \leq k\, Ꝑ_{tcl}(1,Ţ1)^{a}\, Ꝑ_{tcl}(2,Ţ2)^{1-a} = k\, [Ꝑ_{tcl}(1,3)]^{a}[Ꝑ_{tcl}(2,3)]^{1-a} = k\, 1^{a}.\, 1/3^{1-a} \leq k[\, Ꝑ_{tcl}(\sigma,Ţ\sigma)]^{a}\, [Ꝑ_{tcl}(\rho,Ţ\rho)]^{1-a}$$

Case 7

When $\sigma = 1$, $\rho = 3$,

$$Ꝑ_{tcl}(Ţ\sigma,Ţ\rho) = Ꝑ_{tcl}(Ţ1,Ţ3) = Ꝑ_{tcl}(3,3) = 0 \leq k\, Ꝑ_{tcl}(1,Ţ1)^{a}\, Ꝑ_{tcl}(3,Ţ3)^{1-a} = k\, [Ꝑ_{tcl}(1,3)]^{a}[Ꝑ_{tcl}(3,3)]^{1-a} = 0 \leq k[\, Ꝑ_{tcl}(\sigma,Ţ\sigma)]^{a}\, [Ꝑ_{tcl}(\rho,Ţ\rho)]^{1-a}$$

Case 8

When $\sigma = 1$, $\rho = 1$,

$$Ꝑ_{tcl}(Ţ\sigma,Ţ\rho) = Ꝑ_{tcl}(Ţ1,Ţ1) = Ꝑ_{tcl}(3,3) = 0 \leq k\, Ꝑ_{tcl}(1,Ţ1)^{a}\, Ꝑ_{tcl}(1,Ţ1)^{1-a} = k\, [Ꝑ_{tcl}(1,3)]^{a}[Ꝑ_{tcl}(1,3)]^{1-a} = k\, 1^{a}.\, 1^{1-a} \leq k[\, Ꝑ_{tcl}(\sigma,Ţ\sigma)]^{a}\, [Ꝑ_{tcl}(\rho,Ţ\rho)]^{1-a}$$

Case 9

When $\sigma = 2$, $\rho = 0$,

$$Ꝑ_{tcl}(Ţ\sigma,Ţ\rho) = Ꝑ_{tcl}(Ţ2,Ţ0) = Ꝑ_{tcl}(3,3) = 0 \leq k\, Ꝑ_{tcl}(2,Ţ2)^{a}\, Ꝑ_{tcl}(0,Ţ0)^{1-a} = k\, [Ꝑ_{tcl}(2,3)]^{a}[Ꝑ_{tcl}(0,3)]^{1-a} = k\, (1/3)^{a}.\, 3^{1-a} \leq k[\, Ꝑ_{tcl}(\sigma,Ţ\sigma)]^{a}\, [Ꝑ_{tcl}(\rho,Ţ\rho)]^{1-a}$$

Case 10

When $\sigma = 2$, $\rho = 1$,

$$Ꝑ_{tcl}(Ţ\sigma,Ţ\rho) = Ꝑ_{tcl}(Ţ2,Ţ1) = Ꝑ_{tcl}(3,3) = 0 \leq k\, Ꝑ_{tcl}(2,Ţ2)^{a}\, Ꝑ_{tcl}(1,Ţ1)^{1-a} = k\, [Ꝑ_{tcl}(2,3)]^{a}[Ꝑ_{tcl}(1,3)]^{1-a} = k\, (1/3)^{a}.\, 1^{1-a} \leq k[\, Ꝑ_{tcl}(\sigma,Ţ\sigma)]^{a}\, [Ꝑ_{tcl}(\rho,Ţ\rho)]^{1-a}$$

Case 11

When $\sigma = 2$, $\rho = 2$,

$$Ꝑ_{tcl}(Ţ\sigma,Ţ\rho) = Ꝑ_{tcl}(Ţ2,Ţ2) = Ꝑ_{tcl}(3,3) = 0 \leq k\, Ꝑ_{tcl}(2,Ţ2)^{a}\, Ꝑ_{tcl}(2,Ţ2)^{1-a} = k\, [Ꝑ_{tcl}(2,3)]^{a}[Ꝑ_{tcl}(2,3)]^{1-a} = k\, (1/3)^{a}.\, (1/3)^{1-a} \leq k[\, Ꝑ_{tcl}(\sigma,Ţ\sigma)]^{a}\, [Ꝑ_{tcl}(\rho,Ţ\rho)]^{1-a}$$

Case 12

When $\sigma = 2$, $\rho = 3$,

$$Ꝑ_{tcl}(Ţ\sigma,Ţ\rho) = Ꝑ_{tcl}(Ţ2,Ţ3) = Ꝑ_{tcl}(3,3) = 0 \leq k\, Ꝑ_{tcl}(2,Ţ2)^{a}\, Ꝑ_{tcl}(3,Ţ3)^{1-a} = k\, [Ꝑ_{tcl}(2,3)]^{a}[Ꝑ_{tcl}(3,3)]^{1-a} = k\, (1/3)^{a}.\, 0^{1-a} \leq k[\, Ꝑ_{tcl}(\sigma,Ţ\sigma)]^{a}\, [Ꝑ_{tcl}(\rho,Ţ\rho)]^{1-a}$$

Case 13

When $\sigma = 3, \rho = 0$,

$Ꝑ_{tcl}(Ţ\sigma,Ţ\rho) = Ꝑ_{tcl}(Ţ3,Ţ0) = Ꝑ_{tcl}(3,3) = 0 \leq k\, Ꝑ_{tcl}(3,Ţ3)^{a}\, Ꝑ_{tcl}(0,Ţ0)^{1-a} = k\, [Ꝑ_{tcl}(3,3)]^{a}[Ꝑ_{tcl}(0,3)]^{1-a} = k\, 0^{a}.\, 3^{1-a} \leq k[Ꝑ_{tcl}(\sigma,Ţ\sigma)]^{a}\, [Ꝑ_{tcl}(\rho,Ţ\rho)]^{1-a}$

Case 14

When $\sigma = 3, \rho = 1$,

$Ꝑ_{tcl}(Ţ\sigma,Ţ\rho) = Ꝑ_{tcl}(Ţ3,Ţ1) = Ꝑ_{tcl}(3,3) = 0 \leq k\, Ꝑ_{tcl}(3,Ţ3)^{a}\, Ꝑ_{tcl}(1,Ţ1)^{1-a} = k\, [Ꝑ_{tcl}(3,3)]^{a}[Ꝑ_{tcl}(1,3)]^{1-a} = k\, 0^{a}.\, 1^{1-a} \leq k[Ꝑ_{tcl}(\sigma,Ţ\sigma)]^{a}\, [Ꝑ_{tcl}(\rho,Ţ\rho)]^{1-a}$

Case 15

When $\sigma = 3, \rho = 2$,

$Ꝑ_{tcl}(Ţ\sigma,Ţ\rho) = Ꝑ_{tcl}(Ţ3,Ţ3) = Ꝑ_{tcl}(3,3) = 0 \leq k\, Ꝑ_{tcl}(3,Ţ3)^{a}\, Ꝑ_{tcl}(2,Ţ2)^{1-a} = k\, [Ꝑ_{tcl}(3,3)]^{a}[Ꝑ_{tcl}(2,3)]^{1-a} = k\, 0^{a}.\, (1/3)^{1-a} \leq k[Ꝑ_{tcl}(\sigma,Ţ\sigma)]^{a}\, [Ꝑ_{tcl}(\rho,Ţ\rho)]^{1-a}$

Case 16

When $\sigma = 3, \rho = 3$,

$Ꝑ_{tcl}(Ţ\sigma,Ţ\rho) = Ꝑ_{tcl}(Ţ3,Ţ3) = Ꝑ_{tcl}(3,3) = 0 \leq k\, Ꝑ_{tcl}(3,Ţ3)^{a}\, Ꝑ_{tcl}(3,Ţ3)^{1-a} = k\, [Ꝑ_{tcl}(3,3)]^{a}[Ꝑ_{tcl}(3,3)]^{1-a} = k\, 0^{a}.\, 0^{1-a} \leq k[Ꝑ_{tcl}(\sigma,Ţ\sigma)]^{a}\, [Ꝑ_{tcl}(\rho,Ţ\rho)]^{1-a}$

For all $k\, \varepsilon\, (0,1)$, It is clear that all the conditions of theorem 3.1 are satisfied. Therefore , there exists a unique fixed point 3 of Ţ.

Theorem

Let $(Ή,Ꝑ_{tcl})$ be a complete triple controlled metric like space. A self mapping Ţ on a set Ӈ is termed a (k, a,b) – interpolative Kannan contraction. For $\sigma_0\, \varepsilon\, Ӈ$, take $\sigma_n = Ţ^{n}\sigma_0$. Assume that

$$\sup_{m\geq 1}\lim_{i\to\infty} \gamma(\sigma_i, \sigma_{im})\, [\alpha(\sigma_{i+1} , \sigma_{i+2})+ k\, \beta(\sigma_{i+2} , \sigma_{i+3})]/ [\alpha(\sigma_i, \sigma_{i+1})+ k\beta(\sigma_{i+1} , \sigma_{i+2}) < 1/k \qquad 3.18$$

We assume that ,for $\sigma\, \varepsilon\, Ӈ$, we have $\lim_{n\to\infty}\Phi(\sigma_n, \sigma)$, $\lim_{n\to\infty}\Phi(\sigma, \sigma_n)$, $\lim_{n\to\infty}\Phi(\sigma_n, Ţ\sigma)$, $\lim_{n\to\infty}\Phi(Ţ\sigma , \sigma_n)$, $\lim_{n,m\to\infty}\Phi(\sigma_m , \sigma_n)$ where $\Phi\, \varepsilon\, \{ \alpha , \beta ,\gamma \}$, exist and finite for all $n,m\, \varepsilon\, N$, $m\neq n$. .

Then Ţ has a fixed point.

Proof

Define a sequence $\{\sigma_n\}$ as $\sigma_{n+1} = Ţ\sigma_n$ for all $n\, \varepsilon\, N$. Suppose that $\sigma_{n+1} \neq \sigma_n$ for each $n\, \varepsilon\, N$. Thus ,by 3.2, we have

$$Ꝑ_{tcl} (\sigma_n, \sigma_{n+1}) = Ꝑ_{tcl} (Ţ\sigma_{n-1} , Ţ\sigma_n) \leq k\, (Ꝑ_{tcl} (\sigma_{n-1} , Ţ\sigma_{n-1}))^{a}\, (Ꝑ_{tcl} (\sigma_n, Ţ\sigma_n))^{b} = k\, (Ꝑ_{tcl} (\sigma_{n-1} , \sigma_n))^{a}\, (Ꝑ_{tcl} (\sigma_n, \sigma_{n+1}))^{b}$$

$$(Ꝑ_{tcl} (\sigma_n, \sigma_{n+1}))^{1-b} \leq k(Ꝑ_{tcl} (\sigma_{n-1} , \sigma_n))^{a} \qquad 3.19$$

Since a+b< 1, we have

$$Ꝑ_{tcl} (\sigma_n, \sigma_{n+1}) \leq k^{1/1-b}(Ꝑ_{tcl} (\sigma_{n-1} , \sigma_n)) \leq k\, Ꝑ_{tcl} (\sigma_{n-1} , \sigma_n)$$

$$Ᵽ_{tcl}(\sigma_n, \sigma_{n+1}) \leq k\, Ᵽ_{tcl}(\sigma_{n-1}, \sigma_n) \leq k^2\, Ᵽ_{tcl}(\sigma_{n-2}, \sigma_{n-1}) \leq k^3\, Ᵽ_{tcl}(\sigma_{n-3}, \sigma_{n-2}) \ldots \leq k^n\, Ᵽ_{tcl}(\sigma_0, \sigma_1) \quad 3.20$$

As already elaborated in the proof of theorem 3.1, the classical procedure leads to the existence of a fixed point σ^* ε ʜ.

Theorem

Let (Ή,$Ᵽ_{tcl}$) be a complete triple controlled metric like space. A self mapping Ț on a set ʜ is termed a (k, a,b,c) – interpolative Kannan contraction. For σ_0 ε ʜ, take σ_n = $Ț^n\sigma_0$. Assume that

$$\sup_{m\geq 1}\lim_{i\to\infty} \gamma(\sigma_i, \sigma_{im})\,[\alpha(\sigma_{i+1}, \sigma_{i+2})+ k\,\beta(\sigma_{i+2}, \sigma_{i+3})]/\,[\alpha(\sigma_i, \sigma_{i+1})+ k\beta(\sigma_{i+1}, \sigma_{i+2}) < 1/k \quad 3.21$$

We assume that ,for σ ε ʜ , we have $\lim_{n\to\infty}\Phi(\sigma_n, \sigma)$, $\lim_{n\to\infty}\Phi(\sigma, \sigma_n)$, $\lim_{n\to\infty}\Phi(\sigma_n, Ț\sigma)$, $\lim_{n\to\infty}\Phi(Ț\sigma, \sigma_n)$, $\lim_{n,m\to\infty}\Phi(\sigma_m, \sigma_n)$ where Φ ε { α , β ,γ }, exist and finite for all n,m ε N, m≠n.

Then Ț has a fixed point.

Proof

Define a sequence $\{\sigma_n\}$ as σ_{n+1} = Țσ_n for all n ε N. Suppose that $\sigma_{n+1} \neq \sigma_n$ for each n ε N. Thus ,by 3.3, we have

$$Ᵽ_{tcl}(\sigma_n, \sigma_{n+1}) = Ᵽ_{tcl}(Ț\sigma_{n-1}, Ț\sigma_n) \leq k\,(Ᵽ_{tcl}(\sigma_{n-1}, \sigma_n))^a\,(Ᵽ_{tcl}(\sigma_{n-1}, Ț\sigma_{n-1}))^b\,(Ᵽ_{tcl}(\sigma_n, Ț\sigma_n))^c$$

$$= k\,(Ᵽ_{tcl}(\sigma_{n-1}, \sigma_n))^a(Ᵽ_{tcl}(\sigma_{n-1}, \sigma_n))^b(Ᵽ_{tcl}(\sigma_n, \sigma_{n+1}))^c$$

$$(Ᵽ_{tcl}(\sigma_n, \sigma_{n+1}))^{1-c} \leq k(Ᵽ_{tcl}(\sigma_{n-1}, \sigma_n))^{a+b} \quad 3.22$$

Since a+b+c< 1, we have

$$Ᵽ_{tcl}(\sigma_n, \sigma_{n+1}) \leq k^{1/1-c}(Ᵽ_{tcl}(\sigma_{n-1}, \sigma_n)) \leq k\, Ᵽ_{tcl}(\sigma_{n-1}, \sigma_n)$$

$$Ᵽ_{tcl}(\sigma_n, \sigma_{n+1}) \leq k\, Ᵽ_{tcl}(\sigma_{n-1}, \sigma_n) \leq k^2\, Ᵽ_{tcl}(\sigma_{n-2}, \sigma_{n-1}) \leq k^3\, Ᵽ_{tcl}(\sigma_{n-3}, \sigma_{n-2}) \ldots \leq k^n\, Ᵽ_{tcl}(\sigma_0, \sigma_1) \quad 3.23$$

As already elaborated in the proof of theorem 3.1, the classical procedure leads to the existence of a fixed point σ^* ε ʜ.

CONCLUSIONS

In this paper, we explore fixed point results in the context of triple controlled metric-like spaces, building on the foundational results presented in [20]. Our findings extend the work previously documented by providing a more generalized framework that encompasses and elaborates on the fixed point theorems of Kannan and Reich types.

We begin by outlining the properties of triple controlled metric-like spaces and presenting examples that illustrate these concepts. These examples not only support our theoretical results but also demonstrate the practical implications of our findings in mathematical analysis.

Our contribution lies in enhancing the understanding of double controlled metric-like spaces, which play a significant role in various fields of mathematics. By establishing fixed point results within this broader context, we aim to bridge gaps in the existing literature and provide a solid groundwork for further research in this area.

Overall, this work not only generalizes the results of [20] but also opens new avenues for exploration within the framework of metric-like spaces and their applications in mathematical analysis and related disciplines. We encourage further investigation into these spaces and their properties, as well as their implications in other mathematical contexts.

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